

AN IMPROVED LOWER BOUND FOR FINITE ADDITIVE 2-BASES

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ABSTRACT. A set of non-negative integers A is an additive 2-basis with range n , if its sumset $A + A$ contains $0, 1, \dots, n$ but not $n + 1$. Explicit bases are known with arbitrarily large size $|A| = k$ and $n/k^2 \geq 2/7 > 0.2857$. We present a more general construction and improve the lower bound to $85/294 > 0.2891$.

Keywords. Finite additive basis; Additive number theory

1. INTRODUCTION

A set of non-negative integers A is an additive 2-basis of size $k = |A|$ and range $n = n(A)$, if its sumset $A + A$ contains the integers $0, 1, \dots, n$ but not $n + 1$. The maximal ranges

$$n(k) = \max_{|A|=k} n(A)$$

are known up to $n(25) = 212$ [2]. Lacking an explicit formula for $n(k)$, attention has been paid to upper and lower bounds proportional to k^2 . An easy counting argument shows that $n(k) \leq k^2/2 + k/2$. The simple construction $A = \{0, 1, \dots, t, 2t, \dots, t^2\}$ shows that $n(k) \geq k^2/4$.

The upper bound has been improved several times. Yu [5] recently proved that

$$\limsup_{k \rightarrow \infty} \frac{n(k)}{k^2} \leq 0.4585.$$

For the lower bound, an explicit construction by Mrose [4] shows that

$$\liminf_{k \rightarrow \infty} \frac{n(k)}{k^2} \geq 2/7 > 0.2857.$$

Kløve and Mossige [3] presented another construction that achieves the same factor $2/7$. In this note we show that

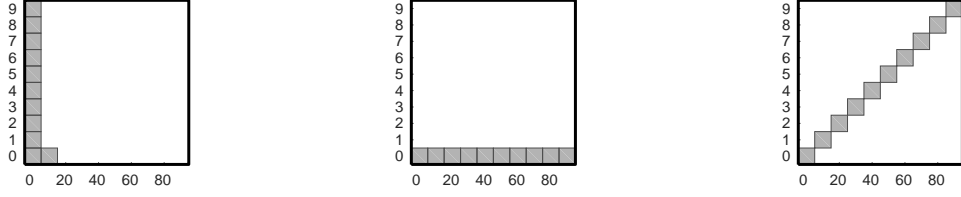
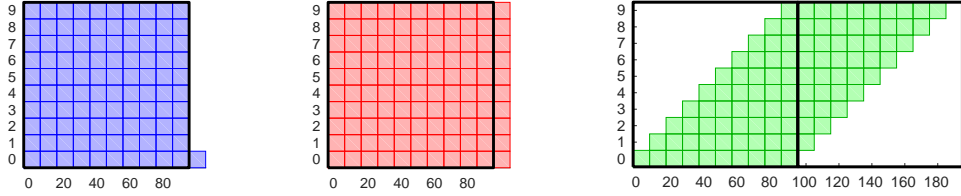
$$(1) \quad \liminf_{k \rightarrow \infty} \frac{n(k)}{k^2} \geq 85/294 > 0.2891.$$

For simplicity we define the size of a basis as $k = |A|$, including the necessary zero element. Often in the literature the zero is not counted, but this makes no difference in the asymptotic ratios.

2. GENERALIZED MROSE BASIS

For finite arithmetic progressions, translation of a set by a constant, and point-wise multiplication we use the notation

$$\begin{aligned} [a, b] &= \{a, a + 1, \dots, b\}, \\ [a, (m), b] &= \{a, a + m, \dots, b\}, \\ A + h &= \{a + h : a \in A\}, \\ h \cdot A &= \{ha : a \in A\}. \end{aligned}$$

FIGURE 1. Three elementary segments: V , H , and S (with $t = 10$).FIGURE 2. Sumsets of elementary segments: $V+H$ (blue), $V+S$ (red), and $H+S$ (green).

Let an integer $t \geq 2$ be given. We will build an additive basis from translated copies of three *elementary segments*:

$$\begin{aligned} V &= [0, t], \\ H &= [0, (t), t^2 - t], \\ S &= [0, (t+1), t^2 - 1]. \end{aligned}$$

Note that $|V| = t + 1$ and $|H| = |S| = t$. It will be beneficial to visualize integers as mapped to planar coordinates by $i \mapsto (\lfloor i/t \rfloor, i \bmod t)$. Then V is a vertical line with an extra element, H is a horizontal line, and S is a slanted line, as illustrated in Figure 1.

The elementary segments give rise to six pairwise sumsets, but $V+V$, $H+H$ and $S+S$ are of negligible size $O(t)$ and will be ignored here. We concentrate on $V+H$, $V+S$, and the parallelogram-shaped $P = H+S$, each of which has size at least t^2 as illustrated in Figure 2. The following facts are easily verified.

Fact 1. Both $V+H$ and $V+S$ contain the square $Q = [0, t^2 - 1]$.

Fact 2. The union of two consecutive parallelograms P and $P+t^2$ contains the square $Q+t^2$.

Fact 3. If elementary segments are translated, their sumsets are likewise translated: for example, $(V+i) + (H+j) = (V+H) + (i+j) \supseteq Q + (i+j)$.

Consider a basis constructed from ℓ elementary segments, placed at specified multiples of t^2 . More precisely, if I, J, K are sets of non-negative integers, with $|I| + |J| + |K| = \ell$, we define

$$(2) \quad A = (V + t^2 \cdot I) \cup (H + t^2 \cdot J) \cup (S + t^2 \cdot K).$$

We say that A is a *generalized Mrose basis* with placement (I, J, K) and segment length t . Using the aforementioned facts we have

$$\begin{aligned} A + A &\supseteq ((V+H) + (I'+J')) \cup ((V+S) + (I'+K')) \cup ((H+S) + (J'+K')) \\ &\supseteq (Q + (I'+J')) \cup (Q + (I'+K')) \cup (P + (J'+K')), \end{aligned}$$

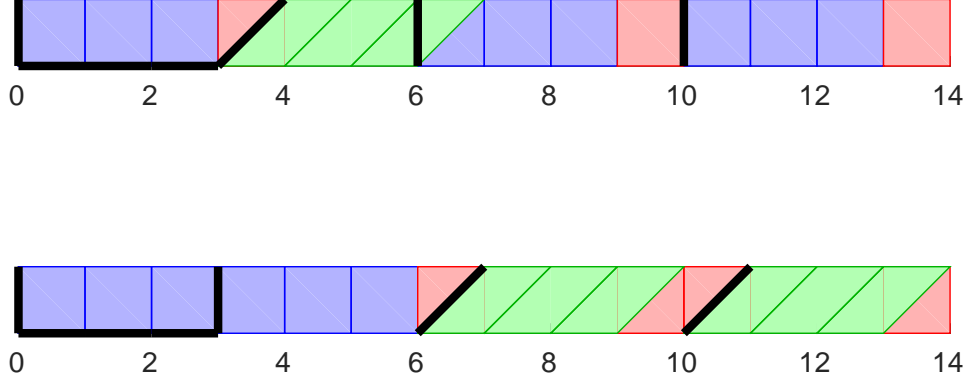


FIGURE 3. Two constructions with $\ell = 7$, $m = 14$, and t large. A unit square represents an interval of t^2 integers. The elementary segments are shown as thick black lines. Copies of $V+H$ and $V+S$ are shown as blue and red squares, respectively. Copies of $H+S$ are shown as green parallelograms. Some squares are only partially visible due to overlap.

where we have written $I' = t^2 \cdot I$ for brevity, and J' , K' likewise. In other words, $A + A$ covers squares at locations $t^2 \cdot (I+J)$ and $t^2 \cdot (I+K)$, and parallelograms at locations $t^2 \cdot (J+K)$.

We now face the combinatorial problem of choosing and placing ℓ copies of elementary segments, so as to maximize the number m of covered consecutive squares beginning from 0. If m such squares are covered, then (2) is an additive basis of size $k \leq \ell(t+1)$ and range

$$n \geq mt^2 - 1 \geq ck^2r - 1,$$

where $c = m/\ell^2$ and $r = (t/(t+1))^2$. The factor r appears because vertical segments have $t+1$ elements, but r tends to 1 as $t \rightarrow \infty$.

Example 1. Choosing $\ell = 2$ and $I = J = \{0\}$, $K = \emptyset$ we have $I+J = \{0\}$, thus $m = 1$ and $c = m/\ell^2 = 1/4$. This is essentially the simple construction mentioned in the introduction.

Example 2. Choosing $\ell = 7$ and $I = \{0, 6, 10\}$, $J = \{0, 1, 2\}$, $K = \{3\}$ gives a basis that is structurally similar to that of Mrose [4], illustrated in the top of Figure 3. The copies of $V+H$ and $V+S$ cover squares $Q + t^2 \cdot \{0, 1, 2, 3, 6, 7, \dots, 13\}$. The copies of $H+S$ cover parallelograms $P + t^2 \cdot \{3, 4, 5\}$, containing in particular the squares $Q + t^2 \cdot \{4, 5\}$. Since $m = 14$ consecutive squares from 0 are covered, we have $c = m/\ell^2 = 2/7$, asymptotically matching Mrose's result.

Example 3. Choosing $\ell = 7$ and $I = \{0, 3\}$, $J = \{0, 1, 2\}$, $K = \{6, 10\}$ we obtain another basis with $m = 14$, illustrated in the bottom of Figure 3. This basis is similar to that of Kløve and Mossige [3].

One can now try different sizes ℓ and placements (I, J, K) , seeking to maximize m/ℓ^2 . With a simple computer program we searched through placements of size $\ell \leq 17$, but found none with $m/\ell^2 > 2/7$. However, from a combination of computer-based search and manual design, we have the following result.

Theorem 1. *There is a placement (I, J, K) with $\ell = 42$ such that $A + A$ covers $m = 510$ consecutive squares beginning from zero.*

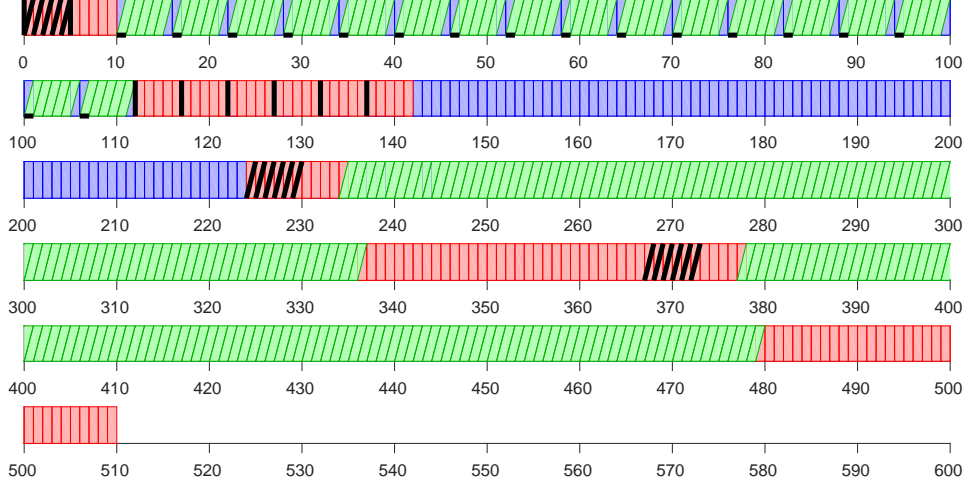


FIGURE 4. A construction of $\ell = 42$ elementary segments. shown as thick black lines. Copies of $V + H$ are shown in blue, copies of $V + S$ in red, and copies of $H + S$ in green (some not visible due to overlap).

Proof. Let

$$I = \{0, 5\} \cup [112, (5), 137],$$

$$J = [10, (6), 106],$$

$$K = [0, 4] \cup [224, 229] \cup [367, 372].$$

Note that $|I| + |J| + |K| = 8 + 17 + 17 = 42$. Let us verify that $2A = A + A$ covers the squares $Q + t^2 \cdot [0, 509]$ as claimed. The proof proceeds by subintervals and is illustrated in Figure 4.

- (i) $Q + t^2 \cdot [0, 9] \subseteq 2A$, since $[0, 9] \subseteq \{0, 5\} + [0, 4] \subseteq I + K$.
- (ii) For each $j \in J$, we observe that $j + [0, 4] \subseteq J + K$, so $2A$ covers consecutive parallelograms $P + t^2 \cdot (j + [0, 4])$, and in particular the squares $Q + t^2 \cdot (j + [1, 4])$. Combining this with the fact that the squares $Q + t^2 \cdot (j + \{0, 5\})$ are covered, it follows that $Q + t^2 \cdot (j + [0, 5])$ is covered. As this holds for all $j \in J$, we see that $Q + t^2 \cdot [10, 111] \subseteq 2A$.
- (iii) $Q + t^2 \cdot [112, 141] \subseteq 2A$, since $[112, 141] \subseteq [112, (5), 137] + [0, 4] \subseteq I + K$.
- (iv) $Q + t^2 \cdot [142, 223] \subseteq 2A$, since $[142, 223] \subseteq [112, (5), 137] + [10, (6), 106] \subseteq I + J$.
- (v) $Q + t^2 \cdot [224, 234] \subseteq 2A$, since $[224, 234] \subseteq \{0, 5\} + [224, 229] \subseteq I + K$.
- (vi) $Q + t^2 \cdot [235, 335] \subseteq P + t^2 \cdot [234, 335]$ by Fact 2. These consecutive parallelograms are covered by $2A$ since $[234, 335] \subseteq [10, (6), 106] + [224, 229] \subseteq J + K$.
- (vii) $Q + t^2 \cdot [336, 366] \subseteq 2A$, since $[336, 366] \subseteq [112, (5), 137] + [224, 229] \subseteq I + K$.
- (viii) $Q + t^2 \cdot [367, 377] \subseteq 2A$, since $[367, 377] \subseteq \{0, 5\} + [367, 372] \subseteq I + K$.
- (ix) $Q + t^2 \cdot [378, 478] \subseteq P + t^2 \cdot [377, 478]$ by Fact 2. These consecutive parallelograms are covered by $2A$ since $[377, 478] \subseteq [10, (6), 106] + [367, 372] \subseteq J + K$.
- (x) $Q + t^2 \cdot [479, 509] \subseteq 2A$, since $[479, 509] \subseteq [112, (5), 137] + [367, 372] \subseteq I + K$.

□

With the placement described above we have

$$c = m/\ell^2 = 510/42^2 = 85/294.$$

In more detail, for any integer $t \geq 2$, this placement gives a generalized Mrose basis of size $k = 42t + 7$ and range $n \geq 510t^2$. We thus have

$$\lim_{t \rightarrow \infty} \frac{n}{k^2} \geq 85/294 > 0.2891,$$

justifying claim (1).

3. DISCUSSION OF FURTHER IMPROVEMENT

Striving for simplicity, we have opted to place the copies of elementary segments at integer multiples of t^2 . It would be possible to move them slightly further: for example, in Mrose's original construction the first segment is $[0, t]$, and the second segment begins at $2t$. However, such changes would in general only extend the range by an amount linear in t , and thus would not improve the asymptotic ratio n/k^2 .

For any additive basis of the form (2), a counting argument provides an upper bound on $|A + A|$ (and hence on $n(A)$). Let the numbers of elementary segments of each kind be $\ell_I = |I|$, $\ell_J = |J|$ and $\ell_K = |K|$. Observing that $V + V$, $H + H$ and $S + S$ have size $O(t)$, and the three “useful” sumsets $V + H$, $V + S$ and $H + S$ have size $t^2 + O(t)$, we have

$$|A + A| \leq (\ell_I \ell_J + \ell_I \ell_K + \ell_J \ell_K) t^2 + O(t).$$

Subject to the constraint $\ell_I + \ell_J + \ell_K = \ell$, we have

$$|A + A| \leq (1/3) \ell^2 t^2 + O(t) = (1/3) k^2 + O(k).$$

In other words, no matter how well the placement I, J, K is chosen, the asymptotic ratio n/k^2 achievable through this construction from three elementary segments cannot essentially exceed $1/3$. If one aims to exceed this ratio, one may want to consider four or more kinds of elementary segments. The challenge is then twofold: first, to design elementary segments with conveniently-shaped sumsets that fit together well; and second, to find a good placement of their copies. This approach could be seen as a decomposition of an additive basis into a “structured part” (elementary segments with fixed structure, but arbitrary size) and an “unstructured part” (placement of segments, perhaps through random or exhaustive search), reminiscent of Bibak's general suggestion [1, p. 114].

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